ON SUMS OF TWO OR FOUR VALUES OF A QUADRATIC FUNCTION OF x*

BY

GORDON PALL

1. We shall consider sums of values of the function

$$q(x) = \mu x^2 + \nu x + c, \tag{1}$$

where $\mu > 0$, ν and c are real. If $q(x)$ is an integer for every integer x, it is of the form

$$\tfrac{1}{2}mx^2 + \tfrac{1}{2}nx + c, \quad m > 0, \tag{2}$$

m, n, and c being integers, $m+n$ even; and conversely.

Denote the least number represented by

$$q(x_1) + q(x_2) + \cdots + q(x_s) \quad (s \text{ given}) \tag{3}$$

for integers $x_i \geqq w$ by $\lambda = \lambda(w, q(x), s)$. In the case (2) our problem may be stated as follows: to determine the magnitude of the largest stretch of consecutive integers $\geqq \lambda$ not represented by (3) for integers $x_i \geqq w$.

We shall obtain a reasonably comprehensive solution of this problem for $s=2$ and $s=4$ (see end of this Section). Many known facts concerning sums of four values of quadratic functions of one variable are corollaries. Thus the Fermat-Cauchy polygonal number theorem is implied by Theorem 2 for the range $-\mu < \nu \leqq -\frac{1}{3}\mu$; for this special case, however, a simpler proof, much like that given here for the range $0 < \mu < \nu$, exists.

§2 is the only one relating to $s=2$. I know of nothing general for $s=3$.

For $s \geqq 5$ Professor L. E. Dickson† gave a complete solution, by ingenious methods depending upon conditions for solving the equations

$$a = x_1^2 + x_2^2 + x_3^2 + x_4^2, \quad b = x_1 + x_2 + x_3 + x_4. \tag{4}$$

The basic lemma, due to Cauchy,‡ is as follows.

LEMMA 1. *Necessary and sufficient conditions that equations* (4) *be solvable in integers* x_i *are*

$$a \equiv b \pmod 2, \quad 4a - b^2 = \textit{a sum of } 3 \textit{ squares}. \tag{5}$$

* Presented to the Society, November 29, 1930; received by the editors May 1, 1931.

† American Journal of Mathematics, vol. 50 (1928), pp. 1–48.

‡ Cf. Legendre, *Théorie des Nombres*, 3d edition, vol. II, nos. 624–30. An outline proof is given here in §4.

If $b \geqq 0$ *these* x_i *are necessarily* $\geqq 0$ *if* (5) *holds and*

$$b^2 + 2b + 4 > 3a. \tag{6}$$

New methods, involving values a, b below the limit $b^2+2b+4>3a$ of Lemma 1, must be invented for all except the simplest cases already done by Professor Dickson.

Let $F_4(x \geqq -k)$, or simply F_k, denote the *table* of all sums of four values of

$$f(x) = \mu x^2 + \nu x \quad (\mu > 0) \tag{7}$$

for integers $x \geqq -k$; or, what is the same thing, the class of all quantities $\mu a + \nu b$, where a and b are integers such that equations (4) are solvable in integers $x_i \geqq -k$. For any μ, ν the *entries* of F_k may be arranged in order of magnitude. By a *gap* in F_k we mean a difference of consecutive entries. If γ_0 is the largest gap for, say, the first thousand entries, we try to show that it is large enough to bridge the entire table. If a larger gap be later encountered it may be taken as a new standard. A difference of entries is called *allowable* if it is $\leqq$ some gap already discovered in the table.

Since sums of four squares or triangular numbers become very numerous for large numbers it might be expected by analogy that the largest gap should always occur early in the tables F_k. In F_∞, in fact, by Theorem 5, the largest gap occurs among the first six entries. In all cases of this sort (Theorems 2, 3, 5, 6, 7, 8) we shall solve our problem completely.

But there are two distinct stretches of values of μ and ν for which the largest gap in F_k occurs arbitrarily far out.

The first case of this is completely solved in Theorem 4. The values μ, ν in question satisfy $\nu < 0$, $\mu \geqq 5|\nu|$.

Finally in the really difficult case with μ, ν in the vicinity of $\mu = (3/2)|\nu|$ the distribution of large gaps is extremely complicated. For every $k > 0$ there are, roughly speaking, infinitely many gaps larger than any near the beginning. I content myself (§§10 and 13) in this case with devising a method for a finite exhaustive determination of the gaps, showing plainly by explicit formulas where the gaps are situated.

Many properties, new and old, are developed of the sum of the roots $\sum x_i$ with $x_i \geqq -k$, when $a = \sum x_i^2$.

2. The case $s = 2$ is trivial, at least for (2).

THEOREM 1. *If, for* $q(x)$ *in* (2), *the values of* $q(x)+q(y)$ *for all integer pairs* x, y *be arranged in order of magnitude, arbitrarily large gaps will occur.*

By (2) the equations

$$N = q(x) + q(y), \tag{8}$$

(9) $$8mN + 2n^2 - 16mc = (2mx + n)^2 + (2my + n)^2$$

are equivalent. Hence (8) is not solvable in integers x, y for r consecutive positive integers N, provided that x^2+y^2 fails to represent $8m(r+1)$ consecutive integers $>2n^2-16\,mc$. The last fact follows from

LEMMA 2. *Any binary quadratic form*

(10) $$\phi = Ax^2 + Bxy + Cy^2,$$

with integer coefficients and discriminant $d=B^2-4AC$ not a square, fails to represent any given number k of consecutive positive integers.

For $\phi \neq N$ if $(d\,|\,p) = -1$ for any odd prime p dividing N to an odd exponent. Since d is not a square there exist infinitely many odd primes p_1, $p_2, \cdots$, such that $(d\,|\,p_i) = -1$. The congruences

$$N \equiv -h \pmod{p_h} \qquad (h = 1, 2, \cdots, k)$$

whose moduli are relative prime in pairs have a solution

$$N = N_0 + lp_1p_2 \cdots p_k,$$

where l is an arbitrary integer. Choosing l so that

$$l \equiv 0 \pmod{p_h} \text{ if } N_0 \not\equiv -h \pmod{p_h^2},$$
$$l \equiv 1 \pmod{p_h} \text{ if } N_0 \equiv -h \pmod{p_h^2},$$

we get integers N such that $N+1, \cdots, N+k$ are not represented by ϕ.

3. The problems for $q(x)$ and $f(x)$ are seen to be equivalent. Also, for any w,

(11) $$\mu x^2 + \nu x = \mu(x - w)^2 + (\nu + 2\mu w)(x - w) + \mu w^2 + \nu w.$$

Hence, by altering the range of x, we can obtain

(12) $$-\mu < \nu \leqq \mu.$$

The following classification is therefore exhaustive:*

(13) $F_0 \equiv F_4(x \geqq 0)$, when $0 < \mu < \nu$;

(14) F_0, when (12) holds;

* It is only for convenience of proof that (14), (15), and (16) have been segregated, since they may be combined as

(13_0) $\mu x^2+\nu x$, $x \geqq x_0$, where $x_0 \leqq 0$ and $-\mu<\nu\leqq\mu$.

By (11) the table for $\mu x^2+\nu x$, $x\geqq x_0$, may be replaced by the table for $\mu x^2+(\nu+2\mu w)x$, $x\geqq x_0-w$. If $\nu+2\mu x_0>\mu$ we choose $w=x_0$; otherwise we can choose a unique integer $w\geqq x_0$ such that $-\mu<\nu+2\mu w\leqq\mu$. In the first case we obtain (13), in the second (13_0).

(15) $$F_k \equiv F_4(x \geqq -k),\ k \geqq 1, \text{ when } |\nu| \leqq \mu < 3|\nu|;$$

(16) $$F_k, \text{ when } \mu \geqq 3|\nu|.$$

4. **Some properties of $\sum x_i$ in $a=\sum x_i^2$.** We use a and e in the rest of this paper to denote positive integers only. For $k\geqq 0$ and any a we define $L_k(a)$ to be the class of all values b such that (4) are solvable in integers $x_i \geqq -k$; but we use it also in the sense of the class of all entries of F_k of which the coefficient of μ is a. We drop the subscript 0 from $L_0(a)$, and define

(17) $$\begin{aligned} B_a &= \text{largest } b \text{ on } L(a); \\ b_a &= \text{least } b \equiv a \pmod 2 \text{ such that } b^2+2b+4>3a; \\ \mathbf{b}_a &= \text{least } b \equiv a \pmod 2 \text{ such that } b^2+4b+16>3a; \\ \mathfrak{b}_k(a) &= \text{least } b \text{ on } L_k(a). \end{aligned}$$

We outline a proof of Lemma 1. We perceive the equivalence of the systems (4) and

(18) $$\begin{aligned} 4a - b^2 &= (x_1+x_2-x_3-x_4)^2+(x_1-x_2+x_3-x_4)^2+(x_1-x_2-x_3+x_4)^2, \\ b &= x_1+x_2+x_3+x_4. \end{aligned}$$

Hence there is a (1,1) correspondence between the sets of integers x_i of (4) and y_h of

(19) $$4a - b^2 = y_2^2 + y_3^2 + y_4^2, \quad b + y_2 + y_3 + y_4 \equiv 0 \pmod 4.$$

For odd a, b, (19_2) holds by choice of sign of one of the (odd) y_k; for given even a, b, (19_2) is a consequence of (19_1). The statement about (5) follows.

As to (6): $\sum x_i \geqq 0$, $x_1<0$ imply

$$\Big(\sum x_i + 1\Big)^2 \leqq (x_2+x_3+x_4)^2 \leqq 3(x_2^2+x_3^2+x_4^2),\ (b+1)^2 \leqq 3(a-1).$$

We do not use here the fact that, when a, b are even, the signs of the y_h are at our disposal, and the preceding can be modified to show that (4) are then solvable in integers $x_i \geqq 0$ if (5_2) holds and merely

(20) $$b \geqq 0,\ 3b^2+8b+16 > 8a.$$

If a, b are odd, $4a-b^2\equiv 3 \pmod 8$; if b is even and $a\equiv 2 \pmod 4$, $a-\frac{1}{4}b^2\equiv 2 \pmod 4$. Hence (5_2) holds provided only that

(21) $$4a \geqq b^2.$$

LEMMA 3. *If e is odd or double an odd, $L_\infty(e)$ consists of every $b\equiv e \pmod 2$ satisfying $4e\geqq b^2$; and $L(e)$ contains every $b\equiv e \pmod 2$ satisfying*

(22) $$b_e \leqq b \leqq B_e.$$

The last part is clear from Lemma 1 and (17). By considering (5_2) we readily verify

LEMMA 4. *Let* $a=4A$, *A odd*, $4a\geqq b^2$. *Then*

$$b=16w \textit{ belongs to } L_\infty(a) \textit{ unless } A\equiv 7 \pmod 8,$$
$$b=4w+2 \textit{ belongs to } L_\infty(a) \textit{ for every } A,$$
$$b=16w+8 \textit{ belongs to } L_\infty(a) \textit{ unless } A\equiv 3 \pmod 8,$$
$$b=4B,\ B \textit{ odd, belongs to } L_\infty(a) \textit{ unless } A-B^2=\Lambda,$$

where Λ *denotes the linear form* $4^q(8v+7)$.*

Now a positive b of the same parity as a satisfies

$$2a^{1/2} \geqq b, \quad b+1 \geqq (3a)^{1/2} \tag{23}$$

for every $a\geqq 1$. By Lemmas 1 and 3 this gives the first part of

LEMMA 5. *If* $a\not\equiv 0 \pmod 4$, *B_a is the maximum* $b\equiv a \pmod 2$ *satisfying* (21). *Also*, $B_a\geqq b_a$. *If a is even*,

$$(B_a)^2 \geqq 3a \tag{24}$$

unless

$$a = 2^{2h-1}A, \quad A = 1, 3, 7, 11, 17, \quad h \geqq 1. \tag{25}$$

In these cases

$$B_a = 2^h z, \quad z = 1, 2, 3, 4, 5, \tag{26}$$

and B_a is the maximum b satisfying (5_2).

Suppose that a is even. Write

$$a = 2^g A, \quad A \text{ odd}, \quad g = 2h \text{ or } 2h-1, \tag{27}$$

so that $h\geqq 1$. If 2^h does not divide b, $4a-b^2=\Lambda$ and is not a sum of three squares. Hence set $b=2^h y$.

The conditions $3a\leqq b^2\leqq 4a$ are

$$3A \leqq y^2 \leqq 4A \quad (\text{if } g \text{ is even}), \tag{28}$$

$$(3/2)A \leqq y^2 \leqq 2A \quad (\text{if } g \text{ is odd}). \tag{29}$$

An odd y satisfies (28) if $(4A)^{1/2}-(3A)^{1/2}\geqq 2$, $A\geqq 57$. An integer y satisfies (29) if $(2A)^{1/2}-(3A/2)^{1/2}\geqq 1$, $A\geqq 29$. By the remarks leading to (21), $4a-b^2$ is then a sum of three squares.

---

* It is of interest to note that there are precisely 52 odd numbers A such that every square $z^2\leqq A$ occurs in the representations of A as a sum of four squares. These are
$A=1$, 3, 5, 9, 13, 17, 21, 25, 33, 41, 45, 49, 57, 65, 73, 81, 89, 97, 105, 129, 145, 153, 169, 177, 185, 201, 209, 217, 225, 257, 273, 297, 305, 313, 329, 345, 353, 385, 425, 433, 441, 481, 513, 561, 585, 609, 689, 697, 713, 817, 825, 945.

Table I is a list* of all values y for which the equations

$$\tau A = \sum x_i^2, \quad 2y = \sum x_i \quad (\tau = 4 \text{ or } 2) \tag{30}$$

are solvable in four integers $x_i \geqq 0$; i.e. $L(a)$ consists of all values $2^h y$. Write

$$\begin{aligned} z &= \text{maximum } y \text{ for a given } a, \\ w &= \text{second largest } y \text{ for a given } a \text{ (if any exists).} \end{aligned} \tag{31}$$

Thus, $B_a = 2^h z$. In the column $y(4A)$ we verify $z^2 \geqq 3A$ if $1 \leqq A \leqq 55$; in the column $y(2A)$ we find $z^2 \geqq (3/2)A$ if $1 \leqq A \leqq 27$, except for $A = 1, 3, 7, 11, 17$, when $z = 1, 2, 3, 4, 5$ respectively. In these five cases z^2 is the largest square $\leqq 2A$.

TABLE I

A	$y(4A)$	$y(2A)$	A	$y(4A)$	$y(2A)$
1	2, 1	1	63	15-11	11-7
3	3	2	65	16-11, 9	11-7
5	4, 3	3, 2	67	16-13, 11	11-8
7	5, 4	3	69	16-11	11-8
9	6, 5, 3	4, 3	71	15–13	11-9
11	6, 5	4	73	17-13, 10	12-7
13	7-5	5-3	75	17-13, 11	12-8
15	7, 6	5, 4	77	17-15, 13, 12	12-8
17	8, 7, 5	5, 4	79	17, 15-12	12-9
19	8, 7	6-4	81	18-15, 13, 9	12-9
21	9-6	6, 5	83	18-14, 11	12-10
23	9, 7	6, 5	85	18-13, 11	13-9, 7
25	10-7, 5	7-4	87	18, 17, 15-13	13-10, 8
27	10-7	7-5	89	18-15, 13	13-8
29	10, 9, 7	7-5	91	19-15, 13, 11	13-9
31	11-9, 7	7, 6	93	19-14, 12	13-9
33	11-8	8-5	95	19-17, 15, 13	13-10
35	11-9	8-6	97	19-16, 14, 13	13-9
37	12-9, 7	8-6	99	19-14	14-10, 8
39	12-9	8, 7	101	20-15, 11	14-9
41	12-9	9-7, 5	103	20-17, 15-13	14-9
43	13, 11, 10, 8	9-6	105	20-15, 13	14-10
45	13-11, 9	9-6	107	19-15	14-10
47	13-10	9-7	109	20, 19, 17-15, 13	14-10
49	14-10, 7	9-7	111	21-17, 15, 12	14-10
51	14, 13, 11, 9	10-8, 6	113	21-15, 13	
53	14-11, 9	10-7	115	21, 19-16, 14	
55	14-11	10-7	117	21-17, 15, 14	
57	15-11, 9	10-8	119	21-17, 15	
59	15-13, 11, 10	10-8	121	22-16, 11	
61	15-11	11-9, 6	123	22, 21, 19-15, 13	
			125	22-15, 13	

* Used also in §8.

LEMMA 6. *For any* $k \geqq 1$ *the largest* b *on* $L_k(a)$ *is* B_a.

When a is odd this is evident from Lemma 5 and the necessity of (21) for (5_2). When a is even it follows from the last clause of Lemma 5 if $(B_a)^2 < 3a$, and, since (6) holds for $b \geqq B_a$, from the last part of Lemma 1 if (24) holds.

LEMMA 7. *Equations* (4) *are solvable in integers* $x_i \geqq -k$ *if the following hold*:

$$(5),\ b \geqq -4k,\ b^2 + 2(k+1)b + 4(k+1)^2 > 3a. \tag{32}$$

This appears out of Lemma 1 if we replace x_i by $x_i - k$ in (4) and obtain the equations

$$a + 2kb + 4k^2 = \sum x_i^2,\ b + 4k = \sum x_i,^*$$

which are to be solvable in $x_i \geqq 0$. (Cf. (17_3).)

LEMMA 8. *For any even* a,

$$B_a \leqq B_{a-1} + 1,\ B_a \leqq B_{a+1} + 1. \tag{33}$$

For, by the maximal property of B_{a-1} (Lemma 5),

$$(B_{a-1} + 2)^2 > 4a \qquad (a \text{ even}). \tag{34}$$

If (33_1) were false we should have

$$B_a \geqq B_{a-1} + 3,\ (B_a)^2 > 4a,$$

contrary to (5_2). Similarly for (33_2) with $a+2$ for a in (34).

LEMMA 9. *For any even a except* (25),

$$b_{a-1} \leqq B_a + 1,\ b_{a+1} \leqq B_a + 1, \tag{35}$$

$$\mathbf{b}_{a-1} \leqq B_a - 1,\ \mathbf{b}_{a+1} \leqq B_a - 1. \tag{36}$$

In fact, by the definitions (17) of b_{a+1}, $\mathbf{b}_{a+1}$,

$$(b_{a+1} - 1)^2 \leqq 3a,\ (\mathbf{b}_{a+1})^2 \leqq 3a - 9 \qquad (a \text{ even}). \tag{37}$$

If (35_2) is false, (37_1) gives

$$b_{a+1} \geqq B_a + 3,\ (B_a)^2 < 3a,$$

contrary to (24). Similarly for (35_1) with $a-2$ for a in (37_1), and for (36) by use of (37_2).

---

* It follows that there exists a (1, 1) correspondence between the sets (a, b) such that (4) is solvable in integers $x_i \geqq 0$, and the sets (a', b') such that (4) is solvable in integers $x_i \geqq -k$. This is defined by

$$a = a' + 2kb' + 4k^2,\quad b' + 4k = b.$$

LEMMA 10. *For any even a,*

$$B_{a+1} - 2 \leqq B_{a-1} \leqq B_{a+1} \qquad (a \text{ even}). \tag{38}$$

Also,

$$B_{a+1} = B_{a-1} + 2 \tag{39}$$

if and only if an odd square lies between $4(a-1)$ *and* $4(a+1)$. *When* (39) *holds,*

$$B_a = B_{a-1} + 1 \text{ or } B_{a+2} = B_{a+1} - 1, \tag{40}$$

according as $a \equiv 2$ *or* $a \equiv 0$ (mod 4).

This is evident from Lemma 5, the parities involved, and the fact that only one odd square can lie between $4a-4$ and $4a+4$.

5. We examine the existence of values $b < b_a$ on $L(a)$. For odd integers $a < 720$ the largest ratio b^2/a, for $b < 2a^{1/2}$ and such that b is missing from $L(a)$, occurs when $a = 347$ and $b = 31$. Then $31 = b_a - 2$. But usually the sequence of b's extends without a break some distance below b_a.

We derive Lemma 11 as a corollary of Lemmas 12 and 13. Lemma 12 is easily proved by the calculus.

LEMMA 11. *If e and x are positive integers,* $e \geqq x^2$, *write*

$$\begin{aligned} g_0(x) &= x + (e - x^2)^{1/2}, \quad g_1(x) = x + (1.8)^{1/2}(e - x^2)^{1/2}, \\ g_2(x) &= x + 3^{1/2}(e - x^2)^{1/2}. \end{aligned} \tag{41}$$

Then, if $e - x^2$ *is a sum of three squares,* $L(e)$ *contains a value b such that*

$$g_0(x) \leqq b \leqq g_2(x).* \tag{42}$$

If also $e - x^2 \not\equiv 1$ (mod 3), $L(e)$ *contains such a value with*

$$g_1(x) \leqq b \leqq g_2(x). \tag{43}$$

LEMMA 12. *Let* ξ, η, ζ *run over all real numbers such that*

$$\xi^2 + \eta^2 + \zeta^2 = c \qquad (c > 0), \tag{44}$$

and (I) $\xi \geqq \eta \geqq \zeta \geqq 0$; (II) $2\eta + 2\zeta \geqq \xi \geqq \eta \geqq \zeta \geqq 0$. *In Case* I *the maximum value of* $\xi + \eta + \zeta$ *is* $(3c)^{1/2}$ *and the minimum value is* $c^{1/2}$; *in Case* II *the minimum value is* $(9c/5)^{1/2}$, *obtained when* $\xi = 2\eta$, $\zeta = 0$.

LEMMA 13. *If* $c \not\equiv 1$ (mod 3) *and c is a sum of three* (*integral*) *squares, then c is of the form*

$$c = X^2 + Y^2 + Z^2, \quad 2Y + 2Z \geqq X \geqq Y \geqq 0, \quad X \geqq Z \geqq 0. \tag{45}$$

* If x is the largest integer such that $e - x^2$ is a sum of three squares, then (42) holds for $b = \mathfrak{b}_0(e)$.

For suppose that

$$c = t^2 + u^2 + v^2, \quad t \geqq u \geqq v \geqq 0, \quad t > 2u + 2v. \tag{46}$$

Then 3 divides at least one of $t+u+v$, $t+u-v$, $t-u+v$, $t-u-v$. In the respective cases write

$$X = \tfrac{1}{3}(2t + 2u - v), \quad Y = \tfrac{1}{3}(2t - u + 2v), \quad Z = \tfrac{1}{3}(t - 2u - 2v);$$
$$X = \tfrac{1}{3}(2t + 2u + v), \quad Y = \tfrac{1}{3}(2t - u - 2v), \quad Z = \tfrac{1}{3}(t - 2u + 2v);$$
$$X = \tfrac{1}{3}(2t + u + 2v), \quad Y = \tfrac{1}{3}(2t - 2u - v), \quad Z = \tfrac{1}{3}(t + 2u - 2v);$$

X and Z = the larger and smaller respectively of

$$\tfrac{1}{3}(2t + u - 2v) \text{ and } \tfrac{1}{3}(t + 2u + 2v),$$

$$Y = \tfrac{1}{3}(2t - 2u + v).$$

We see (1) that X, Y, Z are positive integers; (2) that X is the largest of the three; (3) that $X \leqq 2Y + 2Z$.

The derivatives of the $g_i(x)$ are negative in the interval $.6e^{1/2} \leqq x \leqq e^{1/2}$. Hence the $g_i(x)$ reach their greatest values at the beginning and their least values at the end of any interval

$$\rho e^{1/2} \leqq x \leqq \sigma e^{1/2}, \quad .6 \leqq \rho < \sigma \leqq 1. \tag{47}$$

By Lemma 11 a value b in the interval

$$g_1(\sigma e^{1/2}) \leqq b \leqq g_2(\rho e^{1/2}) \tag{48}$$

will exist on $L(e)$ if e and x satisfy (47) and

$$e - x^2 \not\equiv 1 \pmod 3, \quad e - x^2 \neq \Lambda. \tag{49}$$

We use five pairs (ρ_i, σ_i) $(i = 1, \cdots, 5)$:

$$(.87, .89), (.885, .905), (.915, .935), (.955, .975), (.980, 1). \tag{50}$$

Write $R_i = e^{-1/2} g_1(\sigma_i e^{1/2})$, $S_1 = e^{-1/2} g_2(\rho_i e^{1/2})$. By (41) and (50),

$$\begin{aligned} R_i &\geqq 1.5017, 1.4757, 1.4107, 1.2730, 1; \\ S_i &\leqq 1.7240, 1.6915, 1.6139, 1.4688, 1.3247. \end{aligned} \tag{51}$$

If $e \equiv \pm 1 \pmod 8$, (49) hold for at least one of any q consecutive integers x, where $q = 2, 3, 4, 6$ in the respective cases

$$e \equiv 9 \text{ or } 15, \; e \equiv 7, \; e \equiv 1, \; e \equiv 17 \text{ or } 23 \qquad \pmod{24}. \tag{52}$$

Indeed (49) is true of

any even x if $e \equiv 9$, any odd x if $e \equiv 15 \pmod{24}$,

any $x \equiv 1, 2, 5, 7, 10$, or $11 \pmod{12}$ if $e \equiv 7 \pmod{24}$, any $x \equiv 2, 4, 8,$ or $10 \pmod{12}$ if $e \equiv 1 \pmod{24}$,

any $x \equiv 0 \pmod 6$ if $e \equiv 17$, any $x \equiv 3, 6,$ or $9 \pmod{12}$ if

$e \equiv 23 \pmod{24}$.

Now the interval $\rho_i e^{1/2} \leqq x \leqq \sigma_i e^{1/2}$ contains q integers x if $(.02)^2 e \geqq q^2$, that is,

$$e \geqq 10000,\ 22500,\ 40000,\ 90000 \tag{53}$$

in the various cases (52). Hence

Lemma 14. *For e satisfying* (53) *in the respective cases* (52), $L(e)$ *contains a value $b = b^{(i)}$ satisfying*

$$R_i e^{1/2} \leqq b^{(i)} \leqq S_i e^{1/2} \qquad (i = 1, 2, \cdots, 5), \tag{54}$$

where the R_i and S_i satisfy (51).

6. **Table** $F_4(x \geqq 0)$, $0 < \mu < \nu$. We prove the following theorem:

Theorem 2. *If $\mu > 0$ and $-\mu < \nu$, then*

$$\gamma \equiv \mu + |\nu| \tag{55}$$

is the largest gap in Table F_0.

If $\nu \geqq 0$, γ is the very first gap. Let $\nu < 0$. Then $7\mu + 5\nu$ evidently exceeds every entry of F_0 with $a \leqq 6$. Since $B_a \leqq a - 4$ for every $a \geqq 8$, $8\mu + 4\nu$ is the least entry with $a \geqq 8$. But $8\mu + 4\nu - 7\mu - 5\nu = \mu - \nu$. Hence γ is actually a gap in F_0.

For any $a > 0$ set $\zeta_a = a\mu + B_a \nu$, which is an entry of F_0.

Let $0 < \mu < \nu$. Then $\mu + \nu$, 2μ, and $\nu - \mu$ are allowable differences. Let m be odd and positive. Then (39) holds for $a = e$ and f, where

$$4e = m^2 - 1,\ \ 4f = (m + 2)^2 - 1.$$

By Lemma 10, we can pass from ζ_{e+1} by successive increments 2μ over $\zeta_{e+3}, \cdots$ to ζ_{f-1}. If $f \equiv 2 \pmod 4$ we proceed to ζ_f and ζ_{f+1} by two increments $\mu + \nu$. If $f \equiv 0$ we pass by the increments 2μ, $\mu + \nu$, $\nu - \mu$ to $\omega = \zeta_{f+1} - 2\nu$, ζ_{f+2}, ζ_{f+1}, provided ω is an entry of F_0. This is certainly the case if three integers b lie within the limits (23) for $a = f + 1$; hence, if $f = 20$ and $f \geqq 56$. In the sole remaining case, $4f = 49 - 1$, $f + 1 = 13 = 3^2 + 2^2$, whence $\omega = 13\mu + 5\nu$ is an entry of F_0.

7. **Table** F_0, $\mu \geqq |\nu| > 0$. Now $2|\nu|$ is an allowable difference.

Write $(a, b) = a\mu + b\nu$. If $a \not\equiv 0 \pmod 4$ each of

$$\lambda_a = (a, b_a),\ (a, b_a + 2), \cdots, (a, B_a) = \zeta_a \tag{56}$$

is an entry of F_0, and we can pass allowably between any two such entries for the same a.

By Lemma 8 and the first half of Lemma 9 it is clear then that all gaps in F_0 are allowable except for *the values a in* (25) as follows:

$$(57)\quad \begin{cases} \text{between } (a, B_a) \text{ or } (a-1, B_{a-1}), \text{ and } (a+1, b_{a+1}), \text{ if } \nu > 0; \\ \text{between } (a-1, b_{a-1}) \text{ and either } (a, B_a) \text{ or } (a+1, B_{a+1}) \text{ if } \nu < 0. \end{cases}$$

If $\nu > 0$, let ξ_{a+1} denote an entry $(a+1, x_{a+1})$. Suppose that all gaps in F_0 between ξ_{a+1} and λ_{a+1} are known to be allowable. Then, in further progress from ζ_a to ξ_{a+1} we can suppose $\xi_{a+1} - \zeta_{a-1} > \gamma$, or

$$(58_1)\qquad \mu + |\nu| > (B_{a-1} - x_{a+1} + 2)|\nu|,$$

since otherwise we can pass directly to ξ_{a+1}. The corresponding condition for $\nu < 0$ is

$$(58_2)\qquad \mu + |\nu| > (B_{a+1} - x_{a-1} + 2)|\nu|,$$

where ξ_{a-1} is an entry allowably approachable from λ_{a-1}.

In the cases (52) let e satisfy (53). Write $S_0 = 3^{1/2}$, $b^{(0)} = b_e$. We shall use the preceding with

$$(59)\qquad x_e = b^{(i-1)} \qquad (i = 1, 2, \cdots, 5),$$

where $b^{(i)}$ is any value b on $L(e)$ in (54). By (34), (37_1), (58), and (54), we have

$$(60)\qquad B_f - b^{(i-1)} + 2 \geqq (2 - S_{i-1})e^{1/2} - 1 \quad (e \geqq 3),$$

where

$$(61)\qquad e = a + j,\ f = a - j,$$

and

$$(62)\qquad j = 1 \text{ if } \nu > 0,\ j = -1 \text{ if } \nu < 0.$$

By (54) and $b^{(0)} \leqq S_0 e^{1/2} + 1$,

$$(63)\qquad b^{(i-1)} - b^{(i)} \leqq (2 - S_{i-1})e^{1/2} - 1$$

if $2 \leqq (R_i + 2 - 2S_{i-1})e^{1/2}$, and hence by (51) if $e \geqq 5250$.

Since $(1.4142 - S_5)e^{1/2} \geqq 1$ if $e \geqq 145$,

$$(64)\qquad b^{(5)} \leqq B_a - 1 \text{ if } (B_a)^2 \geqq 2a,$$

for any even $a \geqq 146$. Since $(B_a)^2 \geqq 2a$ in each case (25) we can then pass between ζ_a and $e\mu + b^{(5)}\nu$ by an increment $\leqq \mu - |\nu|$.

(a) $\nu < 0$. There remain the early values $A = 1$ or 3, $h \leqq 7$; $A = 7$, $h \leqq 6$; $A = 11$ or 17, $h \leqq 5$. By calculation, $b_{a-1} = B_a - 1$ if $h = 1$, $A = 1$, 3, 11, or 17;

$b_{a-1}=B_a+1$ if $A=1$, $h=2, 3$; $A=3$, $h=2, 3, 4$; $A=7$, $h=1, 2, 3$; $A=11$ or 17, $h=2, 3, 4, 5$. Finally we give a table for the remaining cases, entries being established by giving x_i satisfying

(65) $$a-1=x_1^2+\cdots+x_4^2, \qquad b=x_1+\cdots+x_4.$$

We give $W=B_{a+1}-b_{a-1}+2$ where necessary.

a	b	x_1	x_2	x_3	x_4	b_{a-1}	W
2^7	17	1	1	5	10	19	4
2^9	31	3	3	3	22	39	8
2^{11}	65	1	1	26	37	79	12
2^{11}	71	1	7	29	34		
2^{13}	129	1	17	26	85	157	26
2^{13}	145	5	14	59	67		
$3\cdot 2^9$	63	6	7	15	35	67	12
$3\cdot 2^{11}$	129	1	27	38	63	135	22
$3\cdot 2^{13}$	257	1	43	102	111	271	44
$7\cdot 2^7$	49	1	7	19	22	51	10
$7\cdot 2^9$	95	10	13	17	55	103	18
$7\cdot 2^{11}$	191	7	37	41	106	207	34

(b) For $\nu>0$ we give an alternative proof for all cases (25), rather than a table for the small cases, which would occupy almost as much space. Let t, u, v have the values 0,1 in

(66) $$(2^{h-1}-t)^2+(2^{h-1}+t)^2+u^2+v^2.$$

For $a=2^{2h-1}$ we thus perceive the entries

(67) $$\zeta_a+r\mu+R\nu, \quad (r, R) = (1, 1), (2, 0), (2, 2), (3, 1), (4, 2).$$

From $14=3^2+2^2+1^2$ we establish at once the entry $\zeta_a+\mu+\nu$ for $A=7$; from $6=2^2+1^2+1^2$, $22=3^2+3^2+2^2$, $34=4^2+3^2+3^2$, we see the entries $\zeta_a+\mu+\nu$, $\zeta_a+2\mu$ for $A=3, 11, 17$.

LEMMA 15. *Let $s=1$ or -1, $0\leqq(2-s)\nu\leqq\mu$, a even, r, R, B integers, $r>0$. Write*

(68) $$\theta=(a+r)\mu+(B+R)\nu.$$

Then

(69) $$\lambda_{a+1}-\max(\theta, \zeta_{a-1})\leqq\mu+s\nu$$

if, for some value $p=0, 1, \cdots, r-1$,

(70) $$(r-p)B_{a-1}+B+R+(2-s)p+1>(r-p+1)(b_{a+1}-s).$$

For, $\eta \equiv \theta - p\{\mu - (2-s)\nu\} \leqq \theta$. Hence (69) holds if $\lambda_{a+1} - \max(\eta, \zeta_{a-1}) \leqq \mu + s\nu$. If $\lambda_{a+1} - \zeta_{a-1} > \mu + s\nu$; i.e. $\mu > (B_{a-1} - b_{a+1} + s)\nu$, then $\lambda_{a+1} - \eta \leqq \mu + s\nu$, or $(r-p)\mu \geqq (b_{a+1} - B - R - (2-s)p - s)\nu$, if (70) holds.

We apply the lemma with $s=1$, $B=B_a$. Then, by (34), (37_1), (25) and (26), (70) follows from

$$(71) \quad a^{1/2}\{2(r - p) + z(2/A)^{1/2} - 3^{1/2}(r - p + 1)\} > 2r - R - 3p - 1.$$

By (25) and (26) the coefficient of $a^{1/2}$ is positive if $r-p>1.2$, .4, .5, .1, .1 respectively. The choices $r=4, 2, 1, 2, 2$, $p=2, 1, 0, 1, 1$ make the right member of (71) zero or negative.

8. **Table** F_k, $k \geqq 1$, $\mu \geqq 3|\nu| > 0$. Since the gap

$$(72) \quad \Gamma \equiv \mu - |\nu|$$

occurs at the beginning of the table, $2|\nu|$ is allowable and part of the treatment is like that of §7. We use $\mathbf{b}_a$ instead of b_a, Γ instead of γ, (36) instead of (35), and see that, in addition to the cases (25), we have to consider the possibilities a even and

$$(73) \quad B_a = B_{a-1} + 1 \text{ if } \nu > 0,\ B_a = B_{a+1} + 1 \text{ if } \nu < 0.$$

When w, of (31), exists,

$$\omega \equiv \omega_a \equiv a\mu + 2^h w\nu \equiv (a, 2^h w)$$

is the entry of $L(a)$ just below ζ_a. Clearly $w \leqq z-1$. Hence, in cases (73_1), $\omega_a - (a-1, B_{a-1}) \leqq \Gamma$; and, in cases ($73_2$), $(a+1, \mathbf{b}_{a+1}) - \omega_a \leqq \Gamma$. If, in these cases,

$$(74) \quad (2^h w)^2 \geqq 3a,$$

then $2^h w > \mathbf{b}_e$, $e = a \mp 1$, by (37_2), and, respectively, $(a+1, \mathbf{b}_{a+1}) - \omega_a \leqq \Gamma$, $\omega_a - (a-1, B_{a-1}) \leqq \Gamma$.

When (73) holds, B_a is, by (34), the largest $b \leqq (2a)^{1/2}$, whence z is the greatest $y \leqq (\tau A)^{1/2}$, where $\tau = 4$ if $g = 2h$, $\tau = 2$ if $g = 2h-1$. Now

$$(75) \quad (z - 2)^2 \geqq 3A \quad (\text{if } g = 2h)$$

if $(4A)^{1/2} - (3A)^{1/2} \geqq 3$, $A \geqq 127$, and

$$(z - 1)^2 \geqq (3/2)A \quad (\text{if } g = 2h - 1)$$

if $(2A)^{1/2} - (3A/2)^{1/2} \geqq 2$, $A \geqq 113$. Hence w exists for these values A, and $(2^h w)^2 \geqq 3a$.

Now (73_1) does not hold if $4(a-1) \geqq (2^h z + 1)^2$, or

$$(76_1) \quad 2^{2h}(\tau A - z^2) \geqq 2^{h+1} z + 5;$$

and (73_2) fails to hold if $4(a+1)\geqq(2^hz+1)^2$, or

$$(76_2)\qquad 2^{2h}(\tau A - z^2) \geqq 2^{h+1}z - 3.$$

Consider column $y(4A)$ of Table I. Observing the entry $f(3)+2f(1)+f(-1)=12\mu+4\nu$ of F_1, we define

$$w = 2 \text{ if } A = 3,\ h = 1,\ g = 2h.$$

We see that $w^2\geqq 3A$ except for

$$(77)\qquad 1 \leqq A \leqq 19,\ A = 23, 29, 35, 41, 43, 71, 79.$$

When $\tau=4$, (76_1) holds in all these cases except

$$(78)\quad h = 1, A = 3, 5, 7;\ h = 1, 2, A = 13, 17;\ h = 1, 2, 3, A = 43;\ A = 1, 9.$$

When $\tau=4$, (76_2) holds in all cases (77) except

$$(79)\quad h = 1, A = 7, 17;\ h = 1, 2, A = 13;\ h = 1, 2, 3, A = 43;\ A = 1, 9.$$

Consider column $y(2A)$ of Table I. In case $h=1$, $A=1$ or 3, we note $2=1^2+(-1)^2$, $6=2^2+(-1)^2+1^2$, and define $w=0$ or 1 respectively. We find $w^2\geqq(3/2)A$ except for

$$(80)\quad 1 \leqq A \leqq 39, 43 \leqq A \leqq 49, A = 55, 57, 59, 67, 69, 71, 81, 83, 97.$$

When $\tau=2$, (76_1) holds in all these cases except

$$(81)\qquad \begin{gathered} h = 1, A = 1, 3, 15, 21, 27, 35, 43, 45, 55; \\ h = 1, 2, A = 5, 9, 19;\ h = 1, 2, 3, A = 13, 25, 33; \end{gathered}$$

in all of which cases w is defined. Also (76_2) holds unless

$$(82)\quad h = 1, A = 9, 27, 35, 43;\ h = 1, 2, A = 5, 19, 33;\ h = 1, 2, 3, A = 13, 25.$$

I. **Remaining cases** (73_1), g **even: namely,** (78). Since w exists, $\omega_a-\alpha\leqq\Gamma$, $\alpha\equiv(a-1,\ B_{a-1})$. Let $\beta=(a+1,\ b_{a+1})$.

(a) $A=3, 5, 7, 13, 17$, $h=1$. Then $z=w+1$.

(b) $A=13, 17$, $h=2$. Then $\omega_a=16A\mu+(A+11)\nu=\beta-\Gamma$.

(c) $A=43$. We observe the entries $\omega_a+\mu\mp\nu$, $\omega_a+2\mu$ since

$$43\cdot 2^{2h} + 1 = (5\cdot 2^h)^2 + (3\cdot 2^h)^2 + (3\cdot 2^h)^2 + (\mp 1)^2,$$
$$43\cdot 2^{2h} + 2 = (5\cdot 2^h)^2 + (3\cdot 2^h + 1)^2 + (3\cdot 2^h - 1)^2 + 0^2.$$

If $h=1$ or 2, $\zeta_a-(\omega_a+2\mu)\leqq 2\nu$, since $\mu\geqq 3\nu$. If $h=3$, $\omega+\mu+\nu=2753\mu+89\nu=\beta$.

(d) $A=1$. Then $\omega=2^{2h}\mu+2^h\nu$, $\alpha=(2^{2h}-1)\mu+(2^{h+1}-1)\nu$. To establish $\omega+\mu\mp\nu$, $\omega+2\mu$, $\omega+3\mu\mp\nu$, $\omega+3\mu+3\nu$ as entries of F_1 give the x_i in $2^{2h}+x_2^2+x_3^2+x_4^2$ values -1, 0, or 1. Finally set $x_2=2$, $x_3=x_4=0$, for the entry $\theta\equiv\omega+4\mu+2\nu$. We use Lemma 15 with $s=-1$, $a=2^{2h}$, $B=2^h$, $r=4$, $R=2$,

$\boldsymbol{b}$ in place of b. When $p=1$, (70) becomes $7\cdot 2^h-1>4\boldsymbol{b}_{a+1}$, and follows from (37_2) for every $h\geqq 1$.

(e) $A=9$. Then $\omega=9\cdot 2^{2h}\mu+5\cdot 2^h\nu$, $\alpha=(9\cdot 2^{2h}-1)\mu+(6\cdot 2^h-1)\nu$. The quantities $\omega+\mu\mp\nu$, $\omega+2\mu$, $\omega+3\mu-\nu$, $\theta\equiv\omega+3\mu+\nu$ are entries of F_1. E.g. if $v=2^{h-1}$,

$$36v^2+3=(4v+1)^2+(4v-1)^2+(2v)^2+(\mp 1)^2.$$

We use Lemma 15 with $s=-1$, $a=9\cdot 2^{2h}$, $B=5\cdot 2^h$, $r=3$, $R=1$, $\boldsymbol{b}$ for b. When $p=0$, (70) is $23\cdot 2^h-5>4\boldsymbol{b}_{a+1}$, which follows from (37_2).

II. **Remaining cases** (73_1), g **odd**; i.e. (81). Again w exists.

(a) $h=1$, all A's in (81). Then $z=w+1$, $\zeta-\omega=2\nu$.

(b) $h=2$, $A=5, 9, 19, 13, 25, 33$. Then $4w=8, 12, 20, 16, 24, 28$. In view of $8A=6^2+2^2, 6^2+6^2, 10^2+6^2+4^2, 8^2+6^2+2^2, 10^2+8^2+6^2, 10^2+10^2+8^2$, the sum of the square roots being $4w$, we have the entries of F_1, $\omega+\mu-\nu$, $\theta\equiv\omega+\mu+\nu$, by adding $(\mp 1)^2$. Since $\zeta-\omega=4\nu$, $\zeta-\theta\leqq 2|\nu|$.

(c) $h=3$, $A=13, 25, 33$. Precisely as in (b), $\theta\equiv\omega+\mu+\nu$ is an entry. Also, $\boldsymbol{b}_{a+1}=35, 47, 55$. Hence $\beta-\theta=2\nu, -2\nu, -2\nu$.

III. **Cases** (25), $\nu>0$. We use Lemma 15 with $s=-1$, $B=B_a$, $\boldsymbol{b}$ for b. Then (70) becomes

$$(r-p)B_{a-1}+B_a+R+4p-r>(r-p+1)\boldsymbol{b}_{a+1},$$

which, by (34), (25), (26), and (37_2), follows from

$$a^{1/2}\{2(r-p)+z(2/A)^{1/2}-3^{1/2}(r-p+1)\}>3r-R-6p. \tag{83}$$

To reobtain the r, R of (67) we need merely interpolate some entries among those exhibited in (b) of §7, by changing some of the $x_i=1$ to $x_i=-1$. We can then evidently reach $\theta=\zeta+r\mu+R\nu$ from ζ by increments $\leqq\Gamma$ or 2ν.

Now (83) is again trivially true, except when $A=7$. Then it becomes

$$a^{1/2}(2+3(2/7)^{1/2}-2\cdot 3^{1/2})>2,\quad a\geqq 202.$$

If $a=14$, $\zeta=14\mu+6\nu=\beta-\Gamma$; if $a=56$, $\theta=57\mu+13\nu=\beta-2\nu$.

Hence we have proved

THEOREM 3. *If* $0<\nu\leqq\frac{1}{3}\mu$, $\Gamma\equiv\mu-|\nu|$ *is the largest gap in* F_1, *and hence in every table* F_k, $k\geqq 1$.

We rework the part of §7 relating to $\nu<0$. The condition corresponding to (58_2) is here

$$\mu-|\nu|>(B_{a+1}-x_{a-1}-2)|\nu|. \tag{84}$$

Taking now $b^0=\boldsymbol{b}_{a-1}$, whence $b^0\leqq S_0e^{1/2}$ by (37_2), we require

$$b^{(i-1)} - b^{(i)} \leqq (2 - S_{i-1})e^{1/2} - 4,$$

which holds if $4 \leqq (R_i+2-2S_{i-1})e^{1/2}$, or $e \geqq 21000$.

IV. **Cases** (25), $\nu<0$. Since $11\cdot 2^{11}>21000$ the same early values remain as in (a) of §7. If $a=2^3, 3\cdot 2^3, 3\cdot 2^5, 7\cdot 2, 7\cdot 2^3, 11\cdot 2^5, 11\cdot 2^7, 17\cdot 2^3, 17\cdot 2^5$, and $17\cdot 2^7$, we find $\boldsymbol{b}_{a-1}=b_{a-1}-2=B_a-1$. For $a=2^7, 2^{11}, 2^{13}, 3\cdot 2^{11}, 3\cdot 2^{13}, 7\cdot 2^7$, we get the entry of $L_1(a-1)$ with $b=B_a-1$ by changing $x_1=1$ to $x_1=-1$ below (65). In addition we have the following entries.

a	b	x_1	x_2	x_3	x_4	$a-1$	$B_{a+1}-\boldsymbol{b}_{a-1}-2$
2^5	7	-1	1	2	5	9	
2^9	35	-1	5	14	17	39	4
$3\cdot 2^7$	31	2	3	9	17	33	4
$7\cdot 2^5$	23	3	3	3	14	25	2

V. **Remaining cases** (73_2), g **odd**: (82).

(a) all $h=1$: $w=z-1$, $\omega-\zeta=2\nu$.

(b) $h=2$, $A=33, 25$: $2^hw=28, 24$; $\boldsymbol{b}_{a-1}=27, 23$.

(c) $h=2$, $A=5, 19, 13$: $2^hw=8, 20, 16$; $\boldsymbol{b}_{a-1}=9, 21, 17$; $39=6^2+1^2+1^2+(-1)^2$, $151=11^2+5^2+2^2+1^2$, $103=7^2+7^2+2^2+(-1)^2$; $b=7, 19, 15$.

(d) $h=3$, $A=25$: $2^hw=48$, $\boldsymbol{b}_{a-1}=47$.

(e) $h=3$, $A=13$: $2^hw=32$, $\boldsymbol{b}_{a-1}=35$, $13\cdot 2^5-1=(\pm 1)^2+5^2+10^2+17^2$; $b=31, 33$.

VI. **All remaining cases except** $a=2^{2h}$, $h\geqq 3$: (79).

(a) $h=1$, $A=7, 17, 13, 43, 1, 9$: $w=z-1$, $\omega-\zeta=2\nu$.

(b) $h=1$, $A=43$: $2^hw=22$, $\boldsymbol{b}_{a-1}=21$.

(c) $h=2$, $A=13$: $2^hw=24$, $\boldsymbol{b}_{a-1}=23$.

(d) $h=2$, $A=43$: $2^hw=44$, $\boldsymbol{b}_{a-1}=45$, $2^4\cdot 43-1=1^2+6^2+17^2+19^2$, $b=43$.

(e) $h=3$, $A=43$: $2^hw=88$, $\boldsymbol{b}_{a-1}=89$, $43\cdot 2^6-1=1^2+15^2+37^2+34^2$, $b=87$.

(f) $A=9$, $g=2h$. By (64) and §7 there remain only $h\leqq 6$. For these we have the following table.

a	2^hw	b	x_1	x_2	x_3	x_4	$\boldsymbol{b}_{a-1}$	B_{a+1}
$9\cdot 2^4$	20	19	-1	5	6	9	19	23
$9\cdot 2^6$	40	39	2	7	9	21	41	47
$9\cdot 2^8$	80	79	5	15	17	42	83	95
$9\cdot 2^{10}$	160	159	15	26	33	85	165	191
$9\cdot 2^{12}$	320	319	35	54	59	171	331	383

(g) $A=1$, $h=2$. $L_1(15)$: 7, 5, 3;

$$L_1(16): 8, 4; \text{ and } 3 < 4.$$

Thus, if $\mu \geqq 3|\nu|$ and $\nu<0$, all gaps in F_1 are $\leqq \Gamma$ except possibly those necessary to pass the points $a=2^{2h}$, $h\geqq 3$.

9. **Gaps associated with $a=2^{2h}$, $k\geqq 1$, $h\geqq 3$, $\nu<0$.** Let $b_{h,k}$ denote the least b on $L_k(2^{2h}-1)$. For any e write $\mathfrak{b}_k(e)$ for the least b on $L_k(e)$. Set

(85) $$\alpha_{h,k} = (2^{2h}-1)\mu + b_{h,k}\nu, \qquad \zeta_h = 2^{2h}\mu + 2^{h+1}\nu,$$ $$\beta_h = (2^{2h}+1)\mu + (2^{h+1}-1)\nu, \quad \omega_h = 2^{2h}\mu + 2^h\nu.$$

If $\mu \geqq |\nu|$, β_h is evidently the least entry of all $L_k(a')$ with $a' \geqq 2^{2h}+1$. We have $\beta_h - \alpha_{h,k} \leqq \Gamma$ if

(86) $$\mu \leqq C_{h,k}|\nu|, \quad C_{h,k} \equiv 2^{h+1} - 2 - b_{h,k}.$$

Suppose that, when $\mu > C_{h,k}|\nu|$, $\alpha_{h,k}$ is the largest entry of all $L_k(a')$ with $a' \leqq 2^{2h}-1$, that is to say,

(87) $$\mathfrak{b}_k(2^{2h}-r) \geqq b_{h,k} - (r-1)C_{h,k} \qquad (r = 2, 3, 4, \cdots).$$

If $\omega_h - \alpha_{h,k} \leqq \Gamma$ we can pass first to ω_h, then to β_h. The contrary case is equivalent to

(88) $$b_{h,k} \geqq 2^h + 1, \text{ or } C_{h,k} \leqq 2^h - 3.$$

If both (87) and (88) hold, F_k contains the gap

(89) $$\Gamma_{h,k} \equiv \min(\beta_h, \omega_h) - \max(\alpha_{h,k}, \zeta_h)$$ $$= \min(2^h|\nu|, \mu + |\nu|, 2\mu - (C_{h,k}+1)|\nu|).$$

The greater of Γ and $\Gamma_{h,k}$ is

(90) $$\Gamma \text{ if } \mu \leqq C_{h,k}|\nu| \text{ or } \mu \geqq (2^h+1)|\nu|,$$ $$2^h|\nu| \text{ if } (2^h-1)|\nu| \leqq \mu \leqq (2^h+1)|\nu|,$$ $$\mu + |\nu| \text{ if } (C_{h,k}+2)|\nu| \leqq \mu \leqq (2^h-1)|\nu|,$$ $$2\mu - (C_{h,k}+1)|\nu| \text{ if } C_{h,k}|\nu| \leqq \mu \leqq (C_{h,k}+2)|\nu|.$$

Since $2^{2h}-1-(2^h-1)^2 \not\equiv 1 \pmod 3$, Lemma 13 shows that

(91) $$b_{h,k} \leqq 2^h - 1 + (9/5)^{1/2}(2^{h+1}-2)^{1/2}.$$

Hence $b_{h,k} < 2^{h+1}-2$, and indeed

(92) $$C_{h,k} \geqq ((2^h-1)^{1/2}-1)^2.$$

We readily find a lower limit for $\mathfrak{b}_k(e)$. Let A denote the greatest integer such that $e - A^2$ is a sum of three squares. If $e > 4k^2$ and $k \geqq 1$,

(93) $$\mathfrak{b}_k(e) \geqq -3k + (e - 3k^2)^{1/2}.$$

Indeed, if $e-A^2>3k^2$, we have

$$\mathfrak{b}_k(e) \geqq A - 2k + (e - A^2 - 2k^2)^{1/2}. \tag{94}$$

Both (93) and (94) can be improved in special cases by various considerations.

If $e=2^{2h}-1$, $A=2^h-1$. Hence by (94), $b_{h,k}>2^h$ as soon as $2^h-1>3k^2$, and possibly for smaller values of h. We find also that (87) holds in virtue of (86_2), (92), (93), and (94), as soon as $2^{h+1}-2>3k^2$. Hence the gap $\Gamma_{h,k}$ occurs in F_k for every h such that $2^h-1>3k^2$, and possibly for smaller values of h. It exceeds Γ only within the range

$$C_{h,k}\,|\,\nu\,| < \mu < (2^h + 1)\,|\,\nu\,| \,. \tag{95}$$

If $b_{h,k}>2^h$ but $2^{h+1}-2<3k^2$ it is necessary, under the present analysis, to verify whether (87) holds. If (87) did not hold, $\Gamma_{h,k}$ would have to be changed by the introduction of new entries in the max term subtracted in (89).

It is easy to see by Lemma 14 and an argument like that employing (84) that we can pass by increments $\leqq \Gamma$ in F_k from $e\mu+\boldsymbol{b}_e\nu$ to $e\mu+b_{h,k}\nu$, where $e=2^{2h}-1$, at least if $h\geqq 8$; and as we shall see, for all e.

(i) $$h = 3.$$

$$L_2(63):15,\ 13,\ 11_1,\ 9_1,\ 7_2;\quad L_2(64):16,\ 8.$$

Here the terms without subscripts belong to $L_0(a)$, and those with subscript j belong to $L_j(a)$ but not to $L_{j-1}(a)$.

Thus $b_{3,1}=9>8$, and F_1 contains the gap $\Gamma_{3,1}\equiv 8\,|\nu|$ if $7\,|\nu|\leqq\mu\leqq 9\,|\nu|$, $2\mu-6\,|\nu|$ if $5\,|\nu|\leqq\mu\leqq 7\,|\nu|$, $\mu+|\nu|$ if $\mu=7\,|\nu|$.

(ii) $$h = 4.$$

$$L_5(256):32,\ 16;$$
$$L_5(255):31,\ 29,\ 27,\ 25,\ 23,\ 21_1,\ 19_2,\ 17_2,\ 13_5,\ 11_5,\ 9_5,\ 7_5,\ 1_5.$$

Hence $b_{4,1}=21$, $b_{4,k}=17$ $(k=2,\ 3,\ 4)$. There is no difficulty in passing from $255\mu+17\nu$ to $255\mu+13\nu$ when $k\geqq 5$, since we can assume $(257\mu+31\nu)-(255\mu+17\nu)>\mu+\nu$. To assure (87) for $k=4$, we verify that

$$\mathfrak{b}_4(254)\geqq 4,\quad \mathfrak{b}_4(253) \geqq -9,\cdots.$$

Since $C_{4,1} = 9$, $C_{4,k} = 13$ $(k = 2, 3, 4)$, the gap (90) is easily written down.

(iii) $$h = 5.$$

$$L_5(1023):63,\cdots,55,\cdots,49,\cdots,39_1,\ 37_3,\ 33_3,\ 31_5.$$

Now $C_{5,1}=C_{5,2}=23$, $C_{5,3}=C_{5,4}=29$.

(iv) $$h = 6.$$

$$L_6(4095): 127, \cdots, 111, \cdots, 97, \cdots, 75_1, 73_2, 71_2, 69_5, 67_5, 63_6.$$

Now $C_{6,1}=51$, $C_{6,2}=C_{6,3}=C_{6,4}=55$, $C_{6,5}=59$.

(v) $$h = 7.$$

$$L_7(16383): 255, \cdots, 221, \cdots, 199, \cdots, 149, 147_2, 145_2, 143_7, 141_6, 139_5, 137_7, 135_5, 131_7, 127_7.$$

Hence $C_{7,1}=105$, $C_{7,2}=C_{7,3}=C_{7,4}=109$, $C_{7,5}=C_{7,6}=119$.

Finally we note that $C_{8,1}=229$. Hence we have

THEOREM 4. *Let* $\mu \geqq -3\nu > 0$, $k \geqq 1$. *Let* $b_{h,k}$ *denote the least* b *on* $L_k(2^{2h}-1)$. *For every* h *such that* $b_{h,k} > 2^h$, *which is true at least if* $2^h - 1 > 3k^2$, F_k *contains a gap just preceding* min (β_h, ω_h) *of* (85) *which exceeds* Γ *for certain values of* μ, ν. *This gap is given in* (90), *with* $C_{h,k}$ *in* (86_2), *if* (87) *holds, which is true if* $2^{h+1}-2 > 3k^2$. *No other gaps in* F_k *exceed* Γ. *In particular the largest gap in* F_1 *is* Γ *if* $3|\nu| \leqq \mu \leqq 5|\nu|$,

$$(96)\qquad \begin{aligned} &\Gamma \textit{ if } (2^{h-1}+1)|\nu| \leqq \mu \leqq C_{h,1}|\nu|, \\ 2\mu - (C_{h,1}+1)|\nu| &\textit{ if } C_{h,1}|\nu| \leqq \mu \leqq (C_{h,1}+2)|\nu|, \\ \mu + |\nu| &\textit{ if } (C_{h,1}+2)|\nu| \leqq \mu \leqq (2^h-1)|\nu|, \\ 2^h|\nu| &\textit{ if } (2^h-1)|\nu| \leqq \mu \leqq (2^h+1)|\nu|, \end{aligned}$$

where $h=3, 4, 5, \cdots$, *and*

$$(97)\quad C_{3,1} = 5,\ C_{4,1} = 9,\ C_{5,1} = 23,\ C_{6,1} = 51,\ C_{7,1} = 105,\ C_{8,1} = 229, \cdots.$$

The largest gap in F_2 *is* Γ *if* $3|\nu| \leqq \mu \leqq 13|\nu|$, *and for the rest is given by* (96) *with* $C_{h,1}$ *replaced by* $C_{h,2}$, $h=4, 5, 6, \cdots$, *and*

$$(98)\qquad C_{4,2} = 13, C_{5,2} = 23, C_{6,2} = 55, C_{7,2} = 109, C_{8,2} \geqq 229, \cdots.$$

The values $C_{h;k}$ *to be used in writing down the largest gaps in* $F_3, \cdots, F_6$, *in the above fashion, are*

$$\begin{aligned} &C_{4,3} = 13, C_{5,3} = 29, C_{6,3} = 55, C_{7,3} = 109, \cdots; \\ &C_{h,4} = C_{h,3} \quad (h = 4, 5, 6, \cdots); \\ &C_{6,5} = 59, C_{7,5} = 119, \cdots; C_{7,6} = 119, \cdots. \end{aligned}$$

If $k \geqq 7$ *all gaps in* F_k *are* $\leqq \Gamma$ *if* $3|\nu| \leqq \mu \leqq 229|\nu|$.

10. **Table** F_k, $|\nu| < \mu < 3|\nu|$, $k \geqq 1$. The writer has previously considered the functions $3x^2 \pm 2x$ (in papers to appear shortly in the Bulletin of the

American Mathematical Society, and the American Journal of Mathematics). The following theorem and three lemmas were proved.

THEOREM 5. *If* $\mu \geqq |\nu| > 0$ *the largest gap in* F_∞ *is*

(99) $$\Gamma \equiv \mu - |\nu| \text{ if } \mu \geqq (3/2)|\nu|, \Delta \equiv 5|\nu| - 3\mu \text{ if } \mu \leqq (3/2)|\nu|.$$

Evidently Γ and Δ are gaps in every F_k. For let $j = (\text{sign } \nu)$ or $1 \cdot (\text{sign } \nu)$. Then they occur from $4f(0)$ to $f(-j)+3f(0)$, and from $4f(-j)$ to $f(j)+3f(0)$.

LEMMA 16. *Although* [by Theorem 5] *every integer* $p \geqq 0$ *is a sum of four values of* $3x^2+2jx$ *for integers* x, *there exist infinitely many integers* $p>0$, *for any* $k \geqq 1$, *such that*

(100) $$p \neq (3x_1^2 + 2jx_1) + \cdots + (3x_4^2 + 2jx_4), \; x_i \geqq -k.$$

LEMMA 17. *If* $k=j=1$ *the only odd* $p>0$ *satisfying* (100) *are*

(101) 9, 13, 25, 29, 41, 45, 47, 69, 75, 79, 97, 109, 149, 165, 189, 235, 305, 509.

If $k=-j=1$ *the only odd* $p>0$ *such that* (100) *holds are*

(102) 33, 59, 129.

Every odd $p>0$ *is a sum of four values* $3x^2+2jx$, $x \geqq -2$.

LEMMA 18. *Let* $k \geqq 0$, $j = \pm 1$. *The only even* $p \geqq 0$ *not sums of four values* $3x^2+2jx$ *for integers* $x \geqq -k$ *are* $\frac{1}{3}(4^h t - 4)$ *where*

(1) t=4, 34, 52, 130, 148, 172, 202, 286, 298, 316, 340, 358, 394, 436, 490, 526, 580, 598, 676, 694, 766, 772, 844, 862, 898, 1102, 1252, 1306; $2^h \not\equiv j \pmod 3$, $2^h > 3k-j$;

(2) t=58, 154, 178, 292, 310, 346, 382, 604, 622, 778, 814, 1006, 1198, 1276, 3676; $2^h \equiv j$, $2 \cdot 2^h > 3k-j$;

(3) t=10, 28, 70, 124, 190, 226, 262, 430, 466; $2^h > 3k-j$, *or* $2^h \equiv j$ *and* $2 \cdot 2^h > 3k-j \geqq 2^h$;

(4) t=94, 244; $2^h > 3k-j$, *or* $2^h \equiv j$ *and* $5 \cdot 2^h > 3k-j \geqq 2^h$;

(5) t=22, 106, 238; $2 \cdot 2h > 3k-j$, *or* $2^h \not\equiv j$ *and* $4 \cdot 2^h > 3k-j \geqq 2 \cdot 2^h$;

(6) t=46, 142; $4 \cdot 2^h > 3k-j$, *or* $2^h \equiv j$ *and* $5 \cdot 2^h > 3k-j \geqq 4 \cdot 2^h$;

(7) t=82, 166, 220, 334; $2^h \not\equiv j$ *and* $4 \cdot 2^h > 3k-j$;

(8) t=76, 484, 652, 1564; $2^h \equiv j$ *and* $5 \cdot 2^h > 3k-j$;

(9) t=508, 1324; $2^h \not\equiv j$ *and* $7 \cdot 2^h > 3k-j$.

It is seen, by continuity, that every F_k contains a gap greater than

(103) $$\epsilon \equiv \max(\Gamma, \Delta)$$

in a neighborhood of $\mu = (3/2)|\nu|$, and that the first such gap will occur as far out as we please for a sufficiently large k.

To determine the least even values p for which (100) holds, write H for the least integer such that $2^H > 3k-j$. If $2^H \equiv j \pmod 3$ the only $4^h t \leqq 6\cdot 4^H$ in Lemma 18 are

(104) $$4^{H-2}t' \qquad (t' = 46, 76, 88, 94).$$

If $2^H \not\equiv j \pmod 3$ the only $4^h t \leqq 3\cdot 4^H$ are

(105) $$4^{H-3}t'' \qquad (t'' = 46^*, 76^*, 88, 94^*, 142^*, 160, 184),$$

where the four starred numbers are to be omitted unless $5\cdot 2^{H-3} > 3k-j$.

Write $M_p = M(p, k, \mu, \nu)$ for the set of all numbers $\mu a + \nu b$ such that

(106) $$p = 3a + 2jb,\ b \text{ on } L_k(a),\ j = \text{sign } \nu.$$

Hence F_k is the ordered class of all elements of all classes M_p, $p = 0, 1, 2, \cdots$. By Lemma 16 infinitely many of the classes M_p are null.

If M_p is not null we write $a_+(p)$ for the largest, $a_-(p)$ for the least a of any element thereof, and $b_+(p)$, $b_-(p)$ for the largest and least values b. Hence

(107) $$p = 3a_+(p) + 2jb_{-j}(p) = 3a_-(p) + 2jb_j(p).$$

By (106) we have

(108) $$a \equiv p \pmod 4,\quad b \equiv -jp \pmod 6.$$

If both $a\mu + b\nu$ and $(a-4)\mu + (b+6j)\nu$ belong to M_p the increment from one to the other of these entries of F_k is allowable if

$$4\mu - 6|\nu| \leqq \mu - |\nu| \text{ and } 6|\nu| - 4\mu \leqq 5|\nu| - 3\mu,$$
$$\text{i.e. if } 3\mu \leqq 5|\nu| \text{ and } |\nu| \leqq \mu.$$

The largest entry of M_p is

(109) $$\xi_+(p) = \mu a_+(p) + \nu b_{-j}(p) \text{ or } \xi_-(p) = \mu a_-(p) + \nu b_j(p)$$

according as

(110) $$\theta \equiv \mu - (3/2)|\nu| \text{ is } \geqq 0 \text{ or } \leqq 0;$$

and the least entry is the other.

In the next two sections we consider completely the cases $k=1$ and 2, and $k \geqq 3$, $\mu \geqq 5|\nu|/3$. We apply the preceding discussion in §13.

11. **Cases $k=1$ and 2, $|\nu| < \mu < 3|\nu|$.** We prove the following theorems:

Theorem 6. *The largest gap in F_1 is*

(111) $$2\nu \text{ if } \nu < \mu < 3\nu,$$

and, if $\nu < 0$, is

$$
(112)\qquad
\begin{aligned}
4|\nu| - 2\mu \; &if\; |\nu| \leqq \mu \leqq (7/5)|\nu|,\\
3\mu - 3|\nu| \; &if\; (7/5)|\nu| \leqq \mu \leqq (3/2)|\nu|,\\
3|\nu| - \mu \; &if\; (3/2)|\nu| \leqq \mu \leqq (5/3)|\nu|,\\
2\mu - 2|\nu| \; &if\; (5/3)|\nu| \leqq \mu \leqq 2|\nu|,\\
2|\nu| \; &if\; 2|\nu| \leqq \mu \leqq 3|\nu|.
\end{aligned}
$$

THEOREM 7. *The largest gap in F_2 is*

$$
(113)\qquad
\begin{aligned}
&5\nu - 3\mu \; if\; \nu \leqq \mu \leqq (4/3)\nu, && 3\mu - 3\nu \; if\; (4/3)\nu \leqq \mu \leqq (7/5)\nu,\\
&4\nu - 2\mu \; if\; (7/5)\nu \leqq \mu \leqq (3/2)\nu, && 2\mu - 2\nu \; if\; (3/2)\nu \leqq \mu \leqq (5/3)\nu,\\
&3\nu - \mu \; if\; (5/3)\nu \leqq \mu \leqq 2\nu, && \mu - \nu \; if\; 2\nu \leqq \mu \leqq 3\nu.
\end{aligned}
$$

If $\nu<0$ the largest gap in this table is

(114)

$$
\begin{aligned}
&5|\nu| - 3\mu \; if\; |\nu| \leqq \mu \leqq (4/3)|\nu|, && 3\mu - 3|\nu| \; if\; (4/3)|\nu| \leqq \mu \leqq (7/5)|\nu|,\\
&4|\nu| - 2\mu \; if\; (7/5)|\nu| \leqq \mu \leqq (5/3)|\nu|, && 4\mu - 6|\nu| \; if\; (5/3)|\nu| \leqq \mu \leqq (7/4)|\nu|,\\
&8|\nu| - 4\mu \; if\; (7/4)|\nu| \leqq \mu \leqq (9/5)|\nu|, && \mu - |\nu| \; if\; (9/5)|\nu| \leqq \mu \leqq 3|\nu|.
\end{aligned}
$$

No entries of F_1 come between $7\mu+3\nu$ and $7\mu+5\nu$ if $\mu>\nu>0$; hence (111) is a gap in F_1. Similarly the gaps (112) for $\nu<0$ occur in the following places: from max $(22\mu+4\nu,\ 26\mu+10\nu)$ to $25\mu+7\nu$ if $|\nu|\leqq\mu\leqq 3|\nu|$, from $15\mu+5\nu$ to min $(17\mu+7\nu,\ 15\mu+3\nu)$ if $(5/3)|\nu|\leqq\mu\leqq 3|\nu|$, from $127\mu+21\nu$ to $125\mu+17\nu$ if $|\nu|\leqq\mu\leqq(3/2)|\nu|$.

In Dickson's table II* multiply the terms free of m by t, and write $m=2\mu$, $t=\mu+\nu$, thereby obtaining table F_1 for $\nu<0$. We easily verify that all gaps in F_1 are $\leqq 4|\nu|-2\mu=m-4t$ or $3\mu-3|\nu|=3t$, at least up to $130\mu+22\nu=54m+22t$. Further, all gaps are $\leqq 3|\nu|-\mu=m-3t$ or $2\mu-2|\nu|=2t$, at least to this point. Finally, all gaps to this point are $\leqq 2|\nu|=m-2t$ if $m\geqq 3t$, i.e. $\mu\leqq 3|\nu|$.

Proof of Theorem 6, by the divisions $L_1(a)$. Suppose first that $\nu>0$. Then (0, 2), (1, −1), and hence (2, −4) are allowable increments. If $e\equiv 1 \pmod 4$ we can pass from (e, B_e) to $(e+1, B_e-1)$, $(e+1, B_{e+1})$, $(e+2, B_{e+1}-1)$, $(e+2, B_{e+2})$ by increments $\mu-\nu$ and 2ν, provided that B_e-2 belongs to $L_1(e)$ if $e\not\equiv 0 \pmod 4$. If $e\equiv 3 \pmod 4$ we pass from (e, B_e) to $(e+2, B_e-4)$, $(e+2, B_e-2)$, $(e+2, B_e)$, $(e+2, B_{e+2})$, provided these quantities belong to $L_1(e+2)$. Finally we may verify

* Bulletin of the American Mathematical Society, vol. 34 (1928), p. 65.

LEMMA 19. *If* $a \not\equiv 0 \pmod 4$, $B_a - 2$ *belongs to* $L_1(a)$ *for every* $a \geqq 1$, *and* $B_a - 4$ *does so except for*

$$a = 1, 5, 9, 13, 14, 23, 29, 49, 71. \tag{115}$$

If $a \equiv 3$ *and* $B_{a+2} = B_a + 2$, $B_a - 4$ *belongs to* $L_1(a+2)$ *unless*

$$a + 2 = 13, 21, 57, 157. \tag{116}$$

Crossing these points is found to introduce no new gaps.

Suppose second that $\nu < 0$. We prove

LEMMA 20. *If* $\mu > |\nu|$, $\nu < 0$, *and* $a \equiv 2 \pmod 4$, *all gaps in* F_1 *are* $\leqq 4|\nu| - 2\mu$ *or* $2\mu - 2|\nu|$ *between* (a, B_a) *and* $(a, B_a - 2)$.

For, if $B_{a+1} = B_a + 1$ we pass from (a, B_a) to $(a+1, B_a+1)$, $(a-1, B_a-3)$, (a, B_a-2). If $B_{a+1} = B_a - 1$, then $B_{a-3} \leqq B_{a-1} = B_a - 1$, and we pass to $(a-3, B_a-5)$, $(a-1, B_a-3)$, (a, B_a-2).

LEMMA 21. *If* $\mu > |\nu|$, $\nu < 0$, *and* $a \equiv 2 \pmod 4$, *we can pass in* F_1 *from* $(a, B_a - 2)$ *to* $(a+4, B_{a+4})$ *by any of the following sets of increments*:

$$\text{I } (-2, -4), (3, 3); \text{ II } (-1, -3), (2, 2); \text{ III } (0, -2), (1, 1), (2, 4).$$

I and II. We pass to $(a+1, B_a-1)$. If $B_{a+3} \geqq B_a + 1$ we proceed to $(a+3, B_a+1)$, $(a+1, B_a-3)$, $(a+3, B_a-1)$, $(a+4, B_a)$. Otherwise, we use $(a-1, B_a-5)$, $(a+1, B_a-3)$, etc.

III. From $(a+1, B_a-1)$ we pass to $(a+1, B_a-3)$, either $(a+3, B_a+1)$ or $(a+1, B_a-5)$, $(a+3, B_a-1)$, $(a+4, B_a)$.

This completes the proof of Theorem 6.

In Dickson's table IV* multiply the terms free of m by t, and write $m = 2\mu$, $t = \mu - \nu$, thus getting table F_2 for $\nu > 0$. The gap $3t$ is seen to occur from $9m - 3t$ to $9m$, if $7t \leqq m$. Now $\Delta = m - 5t$ and $\Gamma = t$. We may verify that all gaps in F_2 are $\leqq m - 5t$ or $2t$ from 0 to $9m - 3t$, and from $9m$ to $198m - 21t = 375\mu + 21\nu$.

If $m \leqq 7t$, i.e. $5\mu \geqq 7\nu$, the gap $4\nu - 2\mu = m - 4t$ occurs in F_2 from $15\mu + 3\nu$ to $13\mu + 7\nu$. If $m \geqq 5t$, i.e. $3\mu \leqq 5\nu$, the gap $2t = 2\mu - 2\nu$ occurs from $15m - 7t$ to $15m - 5t$. If $4t \leqq m \leqq 5t$ the gap $m - 3t = 3\nu - \mu$ occurs from $14m - 2t$ to $15m - 5t$. If $3t \leqq m \leqq 4t$ the gaps $m - 3t$ and $4t - m$ are allowable; we verify that all differences in F_2 at least as far as $198m - 23t = 373\mu + 23\nu$ are $\leqq m - 3t$ or t or $4t - m$.

By this examination, one or the other of the following three sets of increments occurs in F_2 if $\nu \leqq \mu \leqq 3\nu$, and all differences to $373\mu + 23\nu$ are allowable

* Bulletin of the American Mathematical Society, vol. 34 (1928), pp. 210–212.

by any one of these sets: I $(-3, 5)$, $(3, -3)$; II $(-2, 4)$, $(2, -2)$; III $(-1, 3)$, $(1, -1)$, $(2, -4)$.

Let $a \equiv 1 \pmod 4$. We can pass by I or II from (a, B_a) to $(a+2, B_a-2)$, $(a+4, B_a-4)$, either $(a+1, B_a+1)$ or $(a+5, B_a-5)$, $(a+2, B_a)$, $(a+4, B_a-2)$; from (a, B_a-4) to either $(a-3, B_a+1)$ or $(a+1, B_a-5)$, $(a-2, B_a)$, (a, B_a-2), $(a+2, B_a-4)$, $(a+3, B_a-5)$, (a, B_a). By III we can pass from (a, B_a) to $(a+1, B_a-1)$, $(a+2, B_a-2)$, $(a+1, B_{a+1})$, $(a+2, B_{a+1}-1)$, $(a+4, B_{a+1}-5)$, $(a+5, B_{a+1}-6)$, $(a+4, B_{a+1}-3)$, $(a+5, B_{a+1}-4)$, $(a+4, B_{a+1}-1)$, etc. Since we can suppose $a \geqq 350$ we do not need quite all of

LEMMA 22. *If* $e \not\equiv 0 \pmod 4$, B_e-6 *belongs to* $L_2(e)$ *unless*

(117) $$e = 1, 2, 11, 17, 35, 53, 71, 123, 239.$$

If $e \equiv 2 \pmod 4$, B_e-8 *belongs to* $L_2(e)$ *unless*

(118) $$e = 2, 14, 46, 62, 74, 98.$$

In Dickson's table T_2* write $m=2\mu$, $t=\mu+\nu$, obtaining F_2 for $\nu<0$. Again $\Delta=m-5t$ and $\Gamma=t$. The gap $m-4t=4|\nu|-2\mu$ is seen to occur from $53m+21t$ to $54m+17t$ if $6t \leqq m \leqq 7t$, and from $9m+5t$ to $10m+t$ if $5t \leqq m \leqq 6t$. The gap $6t-m$ occurs from $28m+2t$ to $27m+8t$ if $4\frac{2}{3}t \leqq m \leqq 5t$, and the gap $2m-8t$ from $25m+16t$ to $27m+8t$ if $m \leqq 4\frac{2}{3}t$. Finally, $3t$ occurs from $54m+14t$ to $54m+17t$ if $m \geqq 7t$.

The largest of these gaps occurring for the various intervals is shown in (125). Hence the following sets of increments are allowable if $|\nu| \leqq \mu \leqq 3|\nu|$:

(i) $(-3, -5)$, $(2, 2)$ if $\mu \leqq (3/2)|\nu|$;

(ii) $(-2, -4)$, $(1, 1)$, $(4, 6)$ if $(3/2)|\nu| \leqq \mu \leqq (7/4)|\nu|$;

(iii) $(-2, -4)$, $(1, 1)$, $(3, 5)$ if $(5/3)|\nu| \leqq \mu \leqq 2|\nu|$;

(iv) $(-1, -3)$, $(1, 1)$, $(2, 4)$ if $\mu \geqq 2|\nu|$.

An examination of T_2 shows that the gaps (114) are the largest to $54m+17t$, and that we can pass from $54m+17t$ to $144m+36t=324\mu+36\nu$ with differences $m-5t$ and $2t$, or $m-4t$ and t.

Thus we may suppose $a>320$. Then each of B_a, $B_a-2, \cdots, B_a-8$ belongs to $L_2(a)$ if $a \not\equiv 0 \pmod 4$. The passage from (a, B_a-4) to $(a+4, B_{a+4}-4)$ by any of the sets of increments (i), (ii), (iii), (iv) is simple, and left to the reader. It is readily considered on graph paper, with $a \equiv 1 \pmod 4$.

12. **Table** F_k, $k \geqq 3$, $5|\nu| \leqq 3\mu \leqq 9|\nu|$. We prove the following theorem:

* American Journal of Mathematics, loc. cit., pp. 24–25.

THEOREM 8. *The largest gap in* F_3 *is* Γ *if* $\nu<0$ *and* $(5/3)|\nu|\leqq\mu\leqq 3|\nu|$, $4\mu-6\nu$ *if* $(5/3)\nu\leqq\mu\leqq(7/4)\nu$, $8\nu-4\mu$ *if* $(7/4)\nu\leqq\mu\leqq(9/5)\nu$, Γ *if* $(9/5)\nu\leqq\mu\leqq 3\nu$. *The largest in* F_k, $k\geqq 4$, *is* Γ *if* $(5/3)|\nu|\leqq\mu\leqq 3|\nu|$.

If $\mu\geqq 2|\nu|$ this follows from Theorem 7. Let $(5/3)|\nu|\leqq\mu\leqq 2|\nu|$. Then both $(3, -5j)$ and $(-2, 4j)$ are $\leqq\Gamma$, $j=\text{sign }\nu$. If $\nu<0$ the set of gaps (iii) of §11 is allowable, and, by results there obtained, it remains only to examine table F_3 to $54m+17t$. All gaps in T_2 to this point are $\leqq t$ or $m-4t(=4|\nu|-2\mu)$ if we insert the entry $27m+7t$ of F_3.

Hence let $\nu>0$. Denote by π_1 the process of adding $(1, -1)$, $(1, -1)$, $(-2, 4)$ to an entry, and by π_2 that of adding $(1, -1)$, $(1, -1)$, $(3, -5)$, $(1, -1)$, $(-2, 4)$. Let $a\equiv 1 \pmod 4$. The process π_2 and two or three (π_1)'s brings us from (a, B_a) to $(a+4, B_{a+4})$. For this procedure it is necessary that B_a-8 belong to $L_3(a+6)$, B_a-7 to $L_3(a+5)$, $\cdots$.

We find that five consecutive odd values b satisfy

$$b^2+8b+64>3a,\ 4a>b^2,\ b>-12, \tag{119}$$

for any odd a such that

$$273\leqq a\leqq 295,\quad 307\leqq a\leqq 335,\quad 343\leqq a\leqq 377,\quad a\geqq 381;$$

and, permitting ourselves to use the extension of (20_2) analogous to (32_3) for $k=3$, that five consecutive even values b satisfy

$$3b^2+32b+256>8a,\ 4a>b^2,$$

for any $a\equiv 2 \pmod 4$ such that $a\geqq 6$. Since $379=17^2+9^2+3^2$, $29=B_a-8$ belongs to $L_3(a)$, $a=379$. Finally, six consecutive odd values b satisfy (119) if $a=423$, 467, 511, 555, 603, 655, 707, $a\geqq 757$, and B_a-10 belongs to $L_3(a=383)$. No further values $a\equiv 3 \pmod 4$ and >350 satisfy $B_a=B_{a-6}+2$.

It remains only to examine table F_3 to $349\mu+B_{349}\nu$. We insert into table IV (Dickson, loc. cit.) the following entries of F_3:

$$4m+3t,\ 14m-t,\ 18m-t,\ 20m-5t,\ 24m-5t,\ 25m-4t,\ 25m-3t,\ 32m-5t,$$
$$48m-5t,\ 48m-4t,\ 49m-bt\ (b=7, 6, 4, 3),\ 50m-8t,\ 50m-5t.$$

Then all gaps to $50m-10t$ are $\leqq t$, $m-4t$, or $5t-m$, which are our allowable gaps. As in Dickson (American Journal of Mathematics, loc. cit., p. 44), we see that, if $(5/3)\nu\leqq\mu\leqq 2\nu$, F_3 contains a gap from $90\mu+10\nu=50m-10t$ to min $(86\mu+18\nu, 94\mu+4\nu)$. This fact gives the gaps other than Γ in the theorem. From this point to $198m-23t=373\mu+23\nu$ all differences are $\leqq t$, $m-4t$, or $5t-m$. Table F_4 contains the entry $91\mu+9\nu$ which bridges the above gap.

13. **Table** F_k, $k\geqq 3$, $|\nu|<\mu<5|\nu|/3$. By definition of $a_{-j}(q)$ none of

$$(a_v, b_v) \equiv (a_{-j}(q) - 4jv, b_+(q) + 6v) \qquad (v = 1, 2, 3, \cdots),$$

belongs to M_q. It follows that

$$(120) \qquad (b_+(q) + 6)^2 > 4(a_{-j}(q) - 4j) \qquad (q \not\equiv 0, \bmod 4).$$

For, in the contrary case, we must have

$$(121) \qquad b_v^2 + 8b_v + 64 \leqq 3a_v$$

for every $v\geqq 1$ such that $4a_v \geqq b_v^2$. Let V denote the greatest v for which (121) would hold. Then, simultaneously,

$$b_V^2 + 8b_V + 64 \leqq 3a_V, \; 4(a_V - 4j) < (b_V + 6)^2,$$

a contradiction for arbitrary $a_V>0$, $b_V\geqq 3$. It is to be noted that $b_V\geqq b_+(q)+6$, and to be verified that

$$(122) \qquad b_+(q) \geqq -2 - j \qquad (q \not\equiv 0, \bmod 4).$$

Since, then, $a_{-j}(q)$ is as small (if $j=1$) or large (if $j=-1$) as the condition $4a\geqq b^2$ permits, we must have

$$(123) \qquad a_-(q-1) \geqq a_-(q) - 1, \; a_-(q+1) \geqq a_-(q) - 3 \qquad (j = 1),$$

$$(124) \qquad a_+(q+1) \leqq a_+(q) + 1, \; a_+(q-1) \leqq a_+(q) + 3 \qquad (j = -1),$$

provided $q\not\equiv 0$. Some of these are of course vacuous if $M_{q\pm 1}$ is null.

Now, if $3a+2jb+1=3a'+2jb'$, the inequality

$$(125) \qquad (\mu a' + \nu b') - (\mu a + \nu b) \leqq \epsilon$$

is readily seen to be equivalent to

$$(126) \qquad a' \leqq a + 1 \text{ (if } \theta \geqq 0), \; a' \geqq a - 3 \text{ (if } \theta \leqq 0).$$

Hence, if p is even, all differences from the greatest entry β_{p-1} of M_{p-1} to the least entry σ_{p+1} of M_{p+1} are $\leqq \epsilon$ if M_p contains an entry (a^*, b^*) such that

$$(127) \qquad a^* \leqq a_+(p-1) + 1, \; a^* \geqq a_-(p+1) - 1 \; (\theta \geqq 0),$$

$$(128) \qquad a^* \geqq a_-(p-1) - 3, \; a^* \leqq a_+(p+1) + 3 \; (\theta \leqq 0).$$

While all of these hold with $a^*=a_j(p)$ in virtue of (123) and (124) if $p\equiv 2$ (mod 4), generally if $p\equiv 0$ only (127_2) and (128_1) hold if $j=1$, and (127_1) and (128_2) if $j=-1$.

If M_p is null there is always a gap $\gamma_p=\gamma(p, k, \mu, \nu)$ in F_k, from the greatest of the quantities β_{p-r} to the least of the quantities σ_{p+r} $(r=1, 2, 3, \cdots)$. By continuity, as we have seen, this must exceed ϵ for a neighborhood of $\mu=(3/2)\cdot|\nu|$.

If M_p is not null, and $p\equiv 0$, the relations

$$(129)\qquad \begin{aligned} a_-(p) &\geqq a_+(p-1)+5, \quad a_+(p) \leqq a_-(p+1)-5, \\ a_-(p) &\geqq a_+(p+1)+7, \quad a_+(p) \leqq a_-(p-1)-7, \end{aligned}$$

in the respective cases

$$(130)\qquad j=1, \theta \geqq 0; j=-1, \theta \geqq 0; j=1, \theta \leqq 0; j=-1, \theta \leqq 0;$$

are necessary and sufficient conditions for the existence of a gap γ_p in F_k exceeding ϵ for a neighborhood of $\mu=(3/2)|\nu|$. This gap is given by

$$(131)\qquad \begin{aligned} &\min_r \{\xi_{-j}(p), \xi_{-j}(p+r)\} - \max_r \xi_j(p-r), \text{ if } j\theta \geqq 0; \\ &\min_r \xi_j(p+r) - \max_r \{\xi_{-j}(p), \xi_{-j}(p-r)\}, \text{ if } j\theta \leqq 0. \end{aligned}$$

As a further consequence of (120) and of $|\nu|<\mu<5|\nu|/3$, we have that

$$\begin{aligned} \min_r \xi_{-j}(p+r) &= \xi_{-j}(p+1) \text{ if } j\theta \geqq 0, \\ \max_r \xi_{-j}(p-r) &= \xi_{-j}(p-1) \text{ if } j\theta \leqq 0, \end{aligned}$$

which yields a simplification of (131). It is conjectural that

$$(132)\ \xi_j(p-1) = \max \xi_j(p-r) \text{ if } j\theta \geqq 0, \xi_j(p+1) = \min \xi_j(p+r) \text{ if } j\theta \leqq 0,$$

will always hold when

$$(133)\qquad \delta_p > \epsilon,$$

where

$$(134)\qquad \begin{aligned} \delta_p = \delta(p, k, \mu, \nu) &= \xi_{-j}(p+1) - \xi_j(p-1) \text{ if } j\theta \geqq 0, \\ &= \xi_j(p+1) - \xi_{-j}(p-1) \text{ if } j\theta \leqq 0. \end{aligned}$$

Then, if M_p is null, $\gamma_p=\delta_p$; and if M_p is not null, γ_p is the smaller of δ_p and

$$(135)\qquad \xi_{-j}(p) - \xi_j(p-1) \quad (j\theta \geqq 0), \quad \xi_j(p+1) - \xi_{-j}(p) \quad (j\theta \leqq 0).$$

If $3a'+2jb'+2=3a''+2jb''$, then

$$(136)\qquad (\mu a''+\nu b'') - (\mu a'+\nu b') > \epsilon$$

is equivalent to

$$(137)\qquad \begin{aligned} \mu(a'+1-a'') &< \tfrac{1}{2}(3a'+4-3a'')|\nu| \text{ if } \theta \geqq 0, \\ \mu(a''+3-a') &> \tfrac{1}{2}(3a''+8-3a')|\nu| \text{ if } \theta \leqq 0. \end{aligned}$$

If $3a+2b+r-1=3a'+2b'$, $r\geqq 2$, then

$$\mu a + \nu b \leqq \mu a' + \nu b' \tag{138}$$

holds at once if $(a'-a)\theta \geqq 0$, and is a consequence of (137) if

$$\begin{aligned} &a'' < a' < a,\ a + (r-1)a'' \leqq ra' + r - 1 && (\theta \geqq 0); \\ &a < a' \leqq a'' + 2,\ a + (r-1)a'' \geqq ra' - 3r + 3 && (\theta \leqq 0). \end{aligned} \tag{139}$$

If $3a''+2b''+r-1=3a+2b$, $r\geqq 2$, then

$$\mu a + \nu b \geqq \mu a'' + \nu b'' \tag{140}$$

holds at once if $(a-a'')\theta\geqq 0$, and in consequence of (137) if

$$\begin{aligned} &a < a'' < a',\ a + (r-1)a' \geqq ra'' - r + 1 && (\theta \geqq 0); \\ &a > a'' \geqq a' - 2,\ a + (r-1)a' \leqq ra'' + 3r - 3 && (\theta \leqq 0). \end{aligned} \tag{141}$$

These formulas yield sufficient conditions for (132) to hold at least for the values μ, ν satisfying (133), which is of the form of (137).

Now (136) implies $|4n\mu - 6nj\nu| \leqq \epsilon$ if

$$4n \leqq a' + 2 - a''\ (\theta \geqq 0),\ 4n \leqq a'' + 6 - a'\ (\theta \leqq 0). \tag{142}$$

This makes it very probable that *no gaps larger than the greatest of ϵ and the first three or four of the γ_p occur in F_k, for any $k\geqq 3$*, in the range $|\nu|<\mu<5|\nu|/3$ remaining.

To see this let $j=1$, $\theta\geqq 0$ for simplicity. If $p\equiv 2 \pmod 4$ we can always pass from $\xi_-(p-1)$ to $\xi_-(p)$, thence to $\xi_-(p+1)$. This is clear from (123). For the italicized statement above we must be able to pass allowably from $\xi_-(p+1)$ to $\xi_+(p+1)$. Let

$$\alpha_v \equiv \xi_-(p+1) + v(4\mu - 6\nu) \qquad (v = h_1, h_2, \cdots, h_t), \tag{143}$$

where $0\leqq h_1<h_2<\cdots<h_t$, denote the entries of F_k on M_{p+1}. We can pass from α_{h_i} to $\alpha_{h_{i+1}}$ allowably if (by (142)) $h_{i+1}-h_i\leqq h_i$. By Lemma 2 and Theorem 1 of the writer's paper *Improvements of the Cauchy lemma on simultaneous representation*, this is the case at least of $r\equiv 3p+4>10^7$, since then the h_i are distributed in such a way as to satisfy the relation.

Thus we have still to examine the values $r=3p+4$ less than 10^7, a finite though long problem. Enough has been said to indicate the nature of the gaps throughout F_k.

McGill University,
Montreal, Canada